Proofs

Theorem (Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof. We proceed by induction on n. The base case is $n = 0$. (Note this is debatable if you don't agree that $0^0 = 1^1$, which some exam boards do not. Worst case we start with $n = 1$.

Base Case: ($n = 0$).

$$\begin{aligned} LHS &= (x+y)^0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} RHS &= \sum_{k=0}^{0} \binom{0}{k} x^{0-0} y^0 \\ &= \binom{0}{0} x^0 y^0 \\ &= 1 \cdot 1 \cdot 1 = 1 = RHS \end{aligned}$$

Therefore our base case is true.

Inductive step. Suppose that our statement is true for *some* $n = k$, ie $(x+y)^k = \sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r \quad (*)$. Now we must consider $n = k+1$.

$$\begin{aligned} (x+y)^{k+1} &= (x+y)^k (x+y) \\ &= \left(\sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r \right)(x+y) && \text{(by *)} \\ &= \left(\sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r \right) x + \left(\sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r \right) y && \text{(since } a(x+y) = ax + ay \text{, also known as distributivity)} \\ &= \left(\sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r x \right) + \left(\sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r y \right) && \text{(since } (\textstyle\sum x_i) y = \sum x_i y \text{, also distributivity)} \\ &= \left(\sum_{r=0}^{k} \binom{k}{r} x^{k+1-r} y^r \right) + \left(\sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^{r+1} \right) \\ &= \left(\sum_{r=0}^{k} \binom{k}{r} x^{k+1-r} y^r \right) + \left(\sum_{s=1}^{k+1} \binom{k}{s-1} x^{k-(s-1)} y^s \right) && \text{(replacing } r \text{ with } s = r+1 \text{ in the second sum)} \end{aligned}$$

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[1] As Jay-Z put it "If you're havin' math problems I feel bad for you son, I got 99 problems but 0^0 is 1"

$$= \left(\underbrace{\sum_{r=0}^{0}\binom{k}{r}x^{k+1-r}y^{r} \quad + \quad \sum_{r=1}^{k}\binom{k}{r}x^{k+1-r}y^{r}}_{\text{splitting the sum into two parts}} \right) + \left(\sum_{s=1}^{k+1}\binom{k}{s-1}x^{k-(s-1)}y^{s} \right)$$

$$= \binom{k}{0}x^{k+1-0}y^{0} + \sum_{r=1}^{k}\binom{k}{r}x^{k+1-r}y^{r} + \left(\underbrace{\sum_{s=1}^{k}\binom{k}{s-1}x^{k-(s-1)}y^{s} + \sum_{s=k+1}^{k+1}\binom{k}{s-1}x^{k-(s-1)}y^{s}}_{\text{splitting the sum into two parts}} \right)$$

$$= \binom{k}{0}x^{k+1-0}y^{0} + \sum_{r=1}^{k}\binom{k}{r}x^{k+1-r}y^{r} + \sum_{s=1}^{k}\binom{k}{s-1}x^{k-(s-1)}y^{s} + \binom{k}{k}x^{0}y^{k+1}$$

$$= \binom{k}{0}x^{k+1-0}y^{0} + \sum_{r=1}^{k}\binom{k}{r}x^{k+1-r}y^{r} + \underbrace{\sum_{r=1}^{k}\binom{k}{r-1}}_{\text{replacing } s \text{ with } r}x^{k+1-r}y^{r} + \binom{k}{k}x^{0}y^{k+1}$$

$$= \binom{k}{0}x^{k+1-0}y^{0} + \underbrace{\sum_{r=1}^{k}\binom{k}{r}x^{k+1-r}y^{r} + \sum_{r=1}^{k}\binom{k}{r-1}}_{\text{these are summing over the same index}}x^{k+1-r}y^{r} + \binom{k}{k}x^{0}y^{k+1}$$

$$= \binom{k}{0}x^{k+1-0}y^{0} + \quad \underbrace{\sum_{r=1}^{k}\left(\binom{k}{r} + \binom{k}{r-1}\right)x^{k+1-r}y^{r}}_{\text{these are summing over the same index}} \quad + \binom{k}{k}x^{0}y^{k+1}$$

$$= \underbrace{\binom{k+1}{0}}_{=1=\binom{k}{0}}x^{k+1-0}y^{0} + \sum_{r=1}^{k}\binom{k+1}{r}x^{k+1-r}y^{r} + \binom{k}{k}x^{0}y^{k+1} \qquad \text{(Since } \tbinom{k}{r} + \tbinom{k}{r-1} = \tbinom{k+1}{r}\text{)}$$

$$= \sum_{r=0}^{k}\binom{k+1}{r}x^{k+1-r}y^{r} + \underbrace{\binom{k+1}{k+1}}_{=1=\binom{k}{k}}x^{0}y^{k+1} \qquad \text{(reindexing the sum to include the first term)}$$

$$= \sum_{r=0}^{k+1}\binom{k+1}{r}x^{k+1-r}y^{r} \qquad \text{(reindexing the sum to include the last term)}$$

Therefore, if our statement is true for $n = k$ it is true for $n = k + 1$

Since our statement is true for $n = 0$, and if it is true for $n = k$, it is true for $n = k + 1$ by the principle of mathematical induction it is true for $n \geq 0$. □